Fast and Accurate Fourier Series Solutions to Gravitational Lensing by A General Family of Two Power-Law Mass Distributions

Kyu-Hyun Chae

University of Manchester, Jodrell Bank Observatory, Macclesfield, Cheshire SK11 9DL, U.K.; University of Pittsburgh, Department of Physics and Astronomy, Pittsburgh, PA 15260, U.S.A.

ABSTRACT

Fourier series solutions to the deflection and magnification by a family of three-dimensional cusped two power-law ellipsoidal mass distributions are presented. The cusped two power-law ellipsoidal mass distributions are characterized by inner and outer power-law radial indices and a break (or, transition) radius. The model family includes mass models mimicking Jaffe, Hernquist, and η models and dark matter halo profiles from numerical simulations. The Fourier series solutions for the cusped two power-law mass distributions are relatively simple, and allow a very fast calculation even for a chosen small fractional calculational error (e.g. 10^{-5}). These results will be particularly useful for studying lensed systems which provide a number of accurate lensing constraints and for systematic analyses of large numbers of lenses. Subroutines employing these results for the two power-law model and the results by Chae, Khersonsky, & Turnshek for the generalized single power-law mass model are made publicly available.

Subject headings: techniques: analytical — gravitational lensing — galaxies: structure

1. Introduction

Strong gravitational lensing effects provide us with direct tools to probe the potentials of lensing objects (i.e. galaxies and clusters/groups of galaxies) [see, e.g., Schneider, Ehlers, & Falco (1992) for an introduction]. Strong gravitational lensing effects can also be used to constrain cosmological models, in particular to directly measure the Hubble constant (H_0) in a standard cosmology via light travel time differences between lensed images of an extragalactic variable source (Refsdal 1964). Lensed image properties are related to (and

hence probe) the angular structures and radial behaviors of lensing mass distributions.[1] The angular structures and radial behaviors of lensing objects are usually studied using suitable parametric mass models. The choice of a mass model in lensing studies depends on two practical issues, apart from the question of whether the candidate model can be a good approximation to the "true" lensing mass distributions: (1) whether the parameters of the candidate model are constrainable with the available lensing data; (2) the availability of a method which can be used to calculate the deflection and magnification of the candidate model with the speed and accuracy required for the problems under consideration.

This paper is concerned with the second issue for a family of three-dimensional triaxial cusped two power-law mass distributions, which are described by $\rho(R) = \rho_0 R^{-\nu_i}(1 + R^2)^{-(\nu_o - \nu_i)/2}$ $(R^2 = x^2/a^2 + y^2/b^2 + z^2/c^2)$ (Muñoz, Kochanek, & Keeton 2001). This family of mass distributions include mass models resembling well-known galaxy models (e.g. Jaffe 1983, Hernquist 1990, Dehnen 1993, Tremaine et al. 1994, Zhao 1996) and theoretical dark matter halo profiles from numerical simulations (e.g. Navarro, Frenk, & White 1997, Moore et al. 1998). These distributions allow more detailed studies of radial behaviors of galactic mass distributions via a break radius and an outer power-law radial index (i.e. ν_o in the above), compared to the generalized single power-law mass distributions (Chae, Khersonsky, & Turnshek 1998 [hereafter CKT]; Barkana 1998). It is also more realistic than the simpler two-dimensional double power-law models which were used by Evans & Wilkinson (1998) to study analytical properties of cusped lens models. To date, the only available method for calculating the deflection and magnification for this family of three-dimensional mass distributions has been based on numerically evaluating a set of one-dimensional integrals (Muñoz et al. 2001; Keeton 2001). However, the numerical integration method is particularly slow for the cusped two power-law model family because the surface density for the model is not represented by a simple function (see §2). This paper presents mathematical solutions to the deflection and magnification by the cusped two power-law model family. The obtained mathematical solutions allow accurate calculations with errors of 10^{-6} or smaller for most of the parameter space and they improve the calculation speed dramatically for a large portion of the parameter space.

This paper is organized as follows. In §2, we review the mathematical expressions for the projected surface mass densities of the three-dimensional cusped two power-law mass model and the two other models which are necessary to build up the model. In §3, the Fourier series formalism of CKT is reviewed and Fourier series coefficient functions of the three-dimensional cusped two power-law mass model are obtained for two parameter subspaces

---

[1]Gravitational lensing probes the total mass distribution of a lensing object, namely the sum of the luminous and dark mass distributions.

for which the radius is, respectively, smaller and larger than a break radius scaled with an ellipticity; mathematical details are outlined in the Appendices. A discussion of the results is given in §4.

2. The Projected Surface Mass Density for the Cusped Two Power-law Model

The three-dimensional cusped two power-law ellipsoidal mass model, referred to as 'cusp' model, is given by

$$
\rho^{\mathrm{cusp}}(X,Y,Z) = \rho_0 R^{-\nu_i}(1+R^2)^{-(\nu_o-\nu_i)/2} \quad \left[R^2 = \left(\frac{X}{a}\right)^2 + \left(\frac{Y}{b}\right)^2 + \left(\frac{Z}{c}\right)^2 \right] \quad (1)
$$

where ν_i and ν_o are inner and outer power-law radial indices respectively, and $(X,\,Y,\,Z)$ and $(a,\,b,\,c)$ are body coordinates and "break sizes" along the axes, respectively.

The surface mass density for the 'cusp' mass model projected on the lens plane is, up to a multiplicative constant controlling the deflection scale of the lens,[2] given by

$$
\kappa^{\mathrm{cusp}}(\zeta) = \mathrm{B}\left(\frac{1}{2},\frac{\nu_o-1}{2}\right)\frac{1}{(1+\zeta^2)^{(\nu_o-1)/2}}F\left(\frac{\nu_i}{2},\frac{\nu_o-1}{2};\frac{\nu_o}{2};\frac{1}{1+\zeta^2}\right), \quad (2a)
$$

which is, for $\nu_i \neq 1$, equivalent to

$$
\begin{aligned}
\kappa^{\mathrm{cusp}}(\zeta) \;=\; & \mathrm{B}\left(\frac{1}{2},\frac{\nu_i-1}{2}\right)\frac{\zeta^{-(\nu_i-1)}}{(1+\zeta^2)^{(\nu_o-\nu_i)/2}}F\left(\frac{\nu_o-\nu_i}{2},\frac{1}{2};\frac{3-\nu_i}{2};\frac{\zeta^2}{1+\zeta^2}\right) + \\
& \mathrm{B}\left(\frac{\nu_o-1}{2},\frac{1-\nu_i}{2}\right)\frac{1}{(1+\zeta^2)^{(\nu_o-1)/2}}F\left(\frac{\nu_i}{2},\frac{\nu_o-1}{2};\frac{\nu_i+1}{2};\frac{\zeta^2}{1+\zeta^2}\right), \quad (2b)
\end{aligned}
$$

---

[2]The multiplicative constant, denoted by κ_0, is related to the critical radius of the lens for $e=0$, denoted by r_E, by the following equation,

$$
\kappa_0 = \frac{1}{2}\xi^2\left[\mathrm{B}\left(\frac{\nu_o-3}{2},\frac{3-\nu_i}{2}\right)\right]^{-1}J^{-1},
$$

with

$$
\begin{aligned}
J \;=\; & 1 - \left(\frac{1}{1+\xi^2}\right)^{(\nu_o-3)/2}\left[F\left(\frac{\nu_i}{2},\frac{\nu_o-3}{2};\frac{\nu_i-1}{2};\frac{\xi^2}{1+\xi^2}\right)\right. \\
& + \left(\frac{\xi^2}{1+\xi^2}\right)^{(3-\nu_i)/2}\frac{\Gamma(3/2)\Gamma[(\nu_o-\nu_i)/2]\Gamma[(\nu_i-3)/2]}{\Gamma(\nu_i/2)\Gamma[(3-\nu_i)/2]\Gamma[(\nu_o-3)/2]} \\
& \left.\times F\left(\frac{3}{2},\frac{\nu_o-\nu_i}{2};\frac{5-\nu_i}{2};\frac{\xi^2}{1+\xi^2}\right)\right],
\end{aligned}
$$

where $\xi = r_E/r_b$.

where

$$\zeta = \frac{r}{r_b}[1 + e \cos 2(\phi - \phi_0)]^{1/2}, \qquad (3)$$

where parameter $e(\geq 0)$ is related to the ellipticity (ϵ) via $\epsilon = 1 - [(1 - e)/(1 + e)]^{1/2}$. In equations (2a) and (2b), $B(a, b)$ is the beta function and $F(a, b; c; z)$ is the hypergeometric function. The relations between the parameters of the surface mass density [eqs. (2a) & (2b)] and the parameters of the three-dimensional mass density [eq. (1)] and three Eulerian angles can be obtained, e.g., following CKT. The first expression given by equation (2a) is suitable for calculating the deflection and magnification for $r \geq r_b/(1 + e)^{1/2}$, while the second expression given by equation (2b) is suitable for $r < r_b/(1 + e)^{1/2}$.

Each term in (the expansion of) the right-hand side of equation (2a) is the surface mass density of the familiar softened general single power-law mass model (CKT; Barkana 1998) given by

$$\kappa^{\mathrm{sple}}(\zeta; \mu) = \frac{1}{(1 + \zeta^2)^{\mu+1}}, \qquad (4)$$

for which Fourier series solutions for the deflection and magnification are given in CKT. As indicated by a superscript, the surface density of equation (4) is referred to as 'sple' (i.e. softened power-law ellipsoid) model. Equation (2b) consists of terms of the following form, which will be referred to as 'Nuker' model following the *Hubble Space Telescope* group who observed the centers of nearby early-type galaxies (e.g. Lauer et al. 1995), although the inner power-law index [2λ in eq. (5)] can take positive values here,

$$\kappa^{\mathrm{Nuker}}(\zeta; \lambda, \mu) = \frac{\zeta^{2\lambda}}{(1 + \zeta^2)^{\mu+1}}, \qquad (5)$$

where $\lambda > -1$. Note that multiplicative constants were suppressed in both equations (4) and (5) for mathematical convenience, as in equations (2a) and (2b). For positive integer values of λ, equation (5) can be written as

$$\kappa^{\mathrm{Nuker}}(\zeta; \lambda, \mu) = \sum_{l=0}^{\lambda} (-1)^l \binom{\lambda}{l} \kappa^{\mathrm{sple}}(\zeta; \mu - \lambda + l). \qquad (6)$$

Summary of the mass models given above can be found in Table 1.

3. Fourier Series Coefficients for 'CUSP' Model

As was noticed in CKT for the 'sple' model [eq. (4)], the surface potential, deflection and magnification for a surface mass density with elliptical symmetry [i.e., which is a function

of ζ given by equation (3)] can be written as Fourier series with coefficients which can be written as linear combinations of the following "I-functions"[3] of r whose general definitions are given in Appendix A: $I^{(0)}(r)$, $I^{(0')}(r)$,[4] $I_{2m}^{(1)}(r)$, $I_{2m}^{(2)}(r)$, and $I_{2m}^{(3)}(r)$ $(m = 1, 2, 3, \ldots)$.[5] Using these I-functions, we may write the surface potential, $\psi = \psi(r, \phi)$, and its first and second derivatives in polar coordinates as follows:

$$\psi = \int^r I^{(0)}(r')dr' - \sum_{m=1}^{\infty} \frac{r}{2m}[I_{2m}^{(1)}(r) + I_{2m}^{(2)}(r)]\cos[2m(\phi - \phi_0)], \tag{7}$$

$$\frac{\partial \psi}{\partial r} = I^{(0)}(r) + \sum_{m=1}^{\infty}[I_{2m}^{(1)}(r) - I_{2m}^{(2)}(r)]\cos[2m(\phi - \phi_0)], \tag{8}$$

$$\frac{\partial \psi}{r\partial \phi} = \sum_{m=1}^{\infty}[I_{2m}^{(1)}(r) + I_{2m}^{(2)}(r)]\sin[2m(\phi - \phi_0)], \tag{9}$$

$$\frac{\partial}{\partial r}\left(\frac{\partial \psi}{\partial r}\right) = \frac{-I^{(0)}(r) + I^{(0')}(r)}{r}$$
$$-\frac{1}{r}\sum_{m=1}^{\infty}[(2m+1)I_{2m}^{(1)}(r) + (2m-1)I_{2m}^{(2)}(r) - 4I_{2m}^{(3)}(r)]\cos[2m(\phi - \phi_0)], \tag{10}$$

$$\frac{\partial}{r\partial \phi}\left(\frac{\partial \psi}{\partial r}\right) = -\frac{2}{r}\sum_{m=1}^{\infty}m[I_{2m}^{(1)}(r) - I_{2m}^{(2)}(r)]\sin[2m(\phi - \phi_0)], \tag{11}$$

$$\frac{\partial}{\partial r}\left(\frac{\partial \psi}{r\partial \phi}\right) = -\frac{1}{r}\sum_{m=1}^{\infty}[(2m+1)I_{2m}^{(1)}(r) - (2m-1)I_{2m}^{(2)}(r)]\sin[2m(\phi - \phi_0)], \tag{12}$$

$$\frac{\partial}{r\partial \phi}\left(\frac{\partial \psi}{r\partial \phi}\right) = \frac{2}{r}\sum_{m=1}^{\infty}m[I_{2m}^{(1)}(r) + I_{2m}^{(2)}(r)]\cos[2m(\phi - \phi_0)]. \tag{13}$$

As shown above, the m-th order Fourier coefficients of the potential, its first and second derivatives are entirely determined by the m-th order I-functions. For any surface mass density with elliptical symmetry and a reasonable ellipticity, the magnitude of an I-function decreases rapidly as order m increases. The $I^{(1)}(r)$ and $I^{(2)}(r)$ functions are determined solely by the mass distributions within and outside the radius r, respectively. The $I^{(3)}(r)$ functions arise in the derivatives of the $I^{(1)}(r)$ and $I^{(2)}(r)$ functions.

Below we obtain mathematical expressions for the functions $I^{(0)}(r)$, $I^{(0')}(r)$, $I_{2m}^{(1)}(r)$,

---

[3]These I-functions are constants for singular isothermal mass distributions.

[4]This symbol is used throughout in place of $I'^{(0)}(r)$ used in CKT.

[5]For $m = 0$, $I^{(0)}(r) = I_{2m}^{(1)}(r)$ and $I^{(0')}(r) = 2I_{2m}^{(3)}(r)$.

$I_{2m}^{(2)}(r)$, and $I_{2m}^{(3)}(r)$ of the 'cusp' model using the two expressions of the surface density [equations (2a) & (2b)]. As was mentioned in §2, the surface density expressions of the 'cusp' model consist of those of the 'sple' and 'Nuker' models. Thus, in order to calculate the I-functions of 'cusp' model, we need the I-functions of the 'sple' and 'Nuker' models. Mathematical details outlining the derivations of the necessary I-functions of the 'sple' and 'Nuker' models are given in the Appendices A and B.

3.1. The Case $r < r_b/(1+e)^{1/2}$

From equations (2b) and (5), we have

$$
\begin{aligned}
I_{2m}^{\text{cusp}(i)}(r) &= \sum_{n=0}^{\infty}[B_n^{(1)}(\nu_i, \nu_o)I_{2m}^{\text{Nuker}(i)}(r; -(\nu_i - 1)/2 + n, (\nu_o - \nu_i)/2 + n - 1) \\
&\quad + B_n^{(2)}(\nu_i, \nu_o)I_{2m}^{\text{Nuker}(i)}(r; n, (\nu_o - 3)/2 + n)], \\
&\quad (i = 1, 2, 3)
\end{aligned}
\tag{14}
$$

where

$$
B_n^{(1)}(\nu_i, \nu_o) = \text{B}\left(\frac{1}{2}, \frac{\nu_i - 1}{2}\right)\frac{\left(\frac{1}{2}\right)_n \left(\frac{\nu_o - \nu_i}{2}\right)_n}{\left(\frac{3-\nu_i}{2}\right)_n n!},
\tag{15}
$$

$$
B_n^{(2)}(\nu_i, \nu_o) = \text{B}\left(\frac{\nu_o - 1}{2}, \frac{1 - \nu_i}{2}\right)\frac{\left(\frac{\nu_i}{2}\right)_n \left(\frac{\nu_o - 1}{2}\right)_n}{\left(\frac{\nu_i+1}{2}\right)_n n!},
\tag{16}
$$

and $I_{2m}^{\text{Nuker}(i)}(r; \lambda, \mu)$ are the I-functions of the 'Nuker' model [eq. (5)]. In equations (15) and (16) $(a)_b$ is the pochhammer symbol, i.e., $(a)_b = \Gamma(a + b)/\Gamma(a)$. The I-functions of the 'Nuker' model are given below.

a. $\lambda = $ *non-integer*

As outlined in Appendix A, the I-functions of the 'Nuker' model for non-integer values of λ are given by the following simple expressions,

$$
\begin{aligned}
I_{2m}^{\text{Nuker}(i)}(r; \lambda, \mu) &= r[-\sqrt{e_1}]^m \left[\left(\frac{r}{r_b}\right)^2 e_2\right]^{\lambda} \\
&\quad \times \sum_{k=0}^{\infty} C_k^{(i)}(m; \lambda, \mu)\left[\left(\frac{r}{r_b}\right)^2 e_2\right]^k F(m - k - \lambda, -k - \lambda; m + 1; e_1), \\
&\quad (i = 1, 2, 3)
\end{aligned}
\tag{17}
$$

where

$$e_1 = \frac{1 - \sqrt{1 - e^2}}{1 + \sqrt{1 - e^2}}, \tag{18}$$

$$e_2 = \frac{1}{2}(1 + \sqrt{1 - e^2}), \tag{19}$$

$$C_k^{(1)}(m; \lambda, \mu) = \frac{(-1)^k (\mu + 1)_k (\lambda + m + 1)_k (-k - \lambda)_m}{(m + \lambda + 1)(m + \lambda + 2)_k k! m!}, \tag{20}$$

$$C_k^{(2)}(m; \lambda, \mu) = \frac{(-1)^k (\mu + 1)_k (\lambda - m + 1)_k (-k - \lambda)_m}{(m - \lambda - 1)(-m + \lambda + 2)_k k! m!}, \tag{21}$$

$$C_k^{(3)}(m; \lambda, \mu) = \frac{(-1)^k (\mu + 1)_k (-k - \lambda)_m}{k! m!}. \tag{22}$$

b. λ = *zero or positive integer*

For zero or positive integer values of λ, the I-functions can be calculated using equation (6) and the I-functions of the sple model, i.e.

$$I_{2m}^{\text{Nuker}(i)}(r; \lambda, \mu) = \sum_{l=0}^{\lambda} (-1)^l \binom{\lambda}{l} I_{2m}^{\text{sple}(i)}(r; \mu - \lambda + l) \quad (i = 1, 2, 3), \tag{23}$$

where $I_{2m}^{\text{sple}(i)}(r; \mu)$ are the I-functions for the 'sple' model given by equation (4).

3.2. The Case $r \geq r_b/(1 + e)^{1/2}$

For this case, the I-functions are calculated using equation (2a), which may be re-written as

$$\kappa^{\text{cusp}}(\zeta) = \sum_{n=0}^{\infty} D_n(\nu_i, \nu_o) \kappa^{\text{sple}}(\zeta; n + \mu), \tag{24}$$

where $\mu = (\nu_o - 3)/2$, $\kappa^{\text{sple}}(\zeta; \mu)$ is given by equation (4), and

$$D_n(\nu_i, \nu_o) = B\left(\frac{1}{2}, \frac{\nu_o - 1}{2}\right) \frac{\left(\frac{\nu_i}{2}\right)_n \left(\frac{\nu_o - 1}{2}\right)_n}{\left(\frac{\nu_o}{2}\right)_n n!}. \tag{25}$$

Thus, the I-functions of the cusp model may be written as:

$$I_{2m}^{\text{cusp}(i)}(r) = \sum_{n=0}^{\infty} D_n(\nu_i, \nu_o) I_{2m}^{\text{sple}(i)}(r; n + \mu) \quad (i = 1, 2, 3), \tag{26}$$

where $I_{2m}^{\mathrm{sple}(i)}(r;\mu)$ are the I-functions of the 'sple' model [eq. (4)] for $r \geq r_b/(1+e)^{1/2}$ which are given in CKT. The series sum of equation (26) can be efficiently calculated by substituting the series expressions for $I_{2m}^{\mathrm{sple}(i)}(r;\mu)$ given in CKT into the equation, and turning double infinite sums into single infinite sums using the following rule,

$$\sum_{i=0}^{\infty}\sum_{k=0}^{\infty}(\ldots) = \sum_{j=0}^{\infty}\sum_{k=0}^{j}(\ldots), \tag{27}$$

where $j = i + k$.

The following equations (28), (29), (30b), (31), and (32) are obtained from equations (26b), (43), (27b), (28), and (44) of CKT, respectively using the rule given by equation (27), while equation (30a) is obtained from a new expression for $I_{2m}^{\mathrm{sple}(1)}(r;\mu)$ which is given in Appendix B [eq. (B2)]:

$$
\begin{aligned}
I^{\mathrm{cusp}(0)}(r) =\ & \frac{r_b^2}{r}\frac{1}{\sqrt{1-e^2}}\sum_{k=0}^{\infty}\frac{D_k(\nu_i,\nu_o)}{k+\mu} \\
& -h(r)[\varepsilon_2(r)]^{\mu}\sum_{j=0}^{\infty}[\varepsilon_2(r)]^j F(-j-\mu,-j-\mu;1;\varepsilon_1(r))\sum_{k=0}^{j}\frac{D_k(\nu_i,\nu_o)}{k+\mu},
\end{aligned} \tag{28}
$$

$$
I^{\mathrm{cusp}(0')}(r) = 2h(r)[\varepsilon_2(r)]^{\mu}\sum_{j=0}^{\infty}D_j(\nu_i,\nu_o)[\varepsilon_2(r)]^j F(-j-\mu,-j-\mu;1;\varepsilon_1(r)), \tag{29}
$$

$$
\begin{aligned}
I_{2m}^{\mathrm{cusp}(1)}(r) =\ & \frac{r}{\pi}(-te)^m(1-t)^{\mu+1}\sum_{k=0}^{\infty}\frac{[1+(-1)^k](te)^k}{\Gamma(k+m+1)(k+2m+1)}\mathrm{Icos}(k+m,m) \\
& \times \sum_{l=0}^{\infty}D_l(\nu_i,\nu_o)(1-t)^l\frac{\Gamma(l+k+m+\mu+1)}{\Gamma(l+\mu+1)} \\
& \times F(l+k+m+\mu+1,1;k+2m+2;t) \\
& \left(t \equiv \frac{(\frac{r}{r_b})^2}{1+(\frac{r}{r_b})^2};\quad \mathrm{Icos}(k,m) \equiv \int_0^{\pi/2}dx\,\cos^k x\cos mx\right) \\
& [\text{for}\quad r < r_b/(1-e)^{1/2}],
\end{aligned} \tag{30a}
$$

$$
\begin{aligned}
I_{2m}^{\mathrm{cusp}(1)}(r) =\ & -h(r)[-\sqrt{\varepsilon_1(r)}]^m[\varepsilon_2(r)]^{\mu}\frac{1}{\Gamma(m+1)}\sum_{j=0}^{\infty}[\varepsilon_2(r)]^j\frac{\Gamma(m+j+\mu+1)}{\Gamma(-m+j+\mu+1)} \\
& \times F(m-j-\mu,-j-\mu;m+1;\varepsilon_1(r))\sum_{k=0}^{j}D_k(\nu_i,\nu_o)\frac{\Gamma(k+\mu-m)}{\Gamma(k+\mu+1)}
\end{aligned}
$$

$$+h(r)[-\sqrt{\varepsilon_1(r)}\varepsilon_2(r)]^m\frac{1}{\Gamma(m+1)}\left[\sum_{k=0}^{\infty}D_k(\nu_i,\nu_o)\frac{\Gamma(k+\mu-m)}{\Gamma(k+\mu+1)}\right]$$

$$\times\left[\sum_{j=0}^{\infty}[\varepsilon_2(r)]^j\frac{\Gamma(j+2m+1)}{\Gamma(j+1)}F(-j,-j-m;m+1;\varepsilon_1(r))\right]$$

$$[\text{for}\quad r\geq r_b/(1-e)^{1/2}], \tag{30b}$$

$$I_{2m}^{\text{cusp}(2)}(r) = h(r)[-\sqrt{\varepsilon_1(r)}]^m[\varepsilon_2(r)]^\mu\frac{1}{\Gamma(m+1)}$$

$$\times\sum_{j=0}^{\infty}[\varepsilon_2(r)]^j F(m-j-\mu,-j-\mu;m+1;\varepsilon_1(r))$$

$$\times\sum_{k=0}^{j}D_k(\nu_i,\nu_o)\frac{\Gamma(k+m+\mu)}{\Gamma(k+\mu+1)}, \tag{31}$$

$$I_{2m}^{\text{cusp}(3)}(r) = h(r)[-\sqrt{\varepsilon_1(r)}]^m[\varepsilon_2(r)]^\mu\frac{1}{\Gamma(m+1)}$$

$$\times\sum_{j=0}^{\infty}[\varepsilon_2(r)]^j F(m-j-\mu,-j-\mu;m+1;\varepsilon_1(r))$$

$$\times D_j(\nu_i,\nu_o)\frac{\Gamma(j+m+\mu+1)}{\Gamma(j+\mu+1)}. \tag{32}$$

In equations (28), (29), (30b), (31), and (32), the functions $h(r)$, $\varepsilon_1(r)$, and $\varepsilon_2(r)$ are given as follows (CKT):

$$h(r) = \frac{r}{\sqrt{(1+s^2)^2-(es^2)^2}}, \tag{33}$$

$$\varepsilon_1(r) = \left[1-\sqrt{1-\left(\frac{es^2}{1+s^2}\right)^2}\right]\left[1+\sqrt{1-\left(\frac{es^2}{1+s^2}\right)^2}\right]^{-1}, \tag{34}$$

$$\varepsilon_2(r) = \frac{1}{2}\left[\frac{1+s^2}{(1+s^2)^2-(es^2)^2}+\frac{1}{\sqrt{(1+s^2)^2-(es^2)^2}}\right] \tag{35}$$

$$\left(s\equiv\frac{r}{r_b}\right).$$

In equations (28) & (30b), the constant infinite sum can be evaluated using $F(a,b;c;1)=\Gamma(c)\Gamma(c-a-b)/[\Gamma(c-a)\Gamma(c-b)]$ as follows,

$$\sum_{k=0}^{\infty}D_k(\nu_i,\nu_o)\frac{\Gamma(k+\mu-m)}{\Gamma(k+\mu+1)}=\frac{\Gamma(1/2)\Gamma[(\nu_o-3-2m)/2]\Gamma[(3+2m-\nu_i)/2]}{\Gamma[(\nu_o-\nu_i)/2]\Gamma[(3+2m)/2]}, \tag{36}$$

where m is zero or positive integer. Equation (30a) converges in a non-trivial fashion and requires a careful treatment. The convergence property of equation (30a) and a strategy for its efficient numerical evaluation are discussed in Appendix C.

4.   Discussion

We have obtained Fourier series solutions to the deflection and magnification by the three-dimensional cusped two power-law ellipsoidal mass ('cusp') model [eq. (1) or eqs. (2a,b); see also Muñoz et al. (2001)] by calculating the I-functions of the model whose general definitions are given in Appendix A. The I-functions of the 'cusp' model given in §3 are well-defined rapidly converging series and can be evaluated efficiently using recursion relations for gamma functions and those for hypergeometric functions for the successive terms in the series (Abramowitz & Stegun 1964; Gradshteyn & Ryzhik 1994). Using these I-functions, the deflection and magnification can be calculated quickly and accurately for arbitrary parameter values. In particular, for any value of the radius sufficiently smaller or larger than the break radius the calculation is very fast, even for highly accurate calculation, e.g., with fractional errors of 10^{-5} or smaller. For example, compared with that for the 'sple' model for a given value of the normalized radius r/r_b, the calculational speed for the 'cusp' model is slower by factors of ~ 3 and ~ 8 respectively for $r/r_b = 10$ and 0.1.[6] It typically takes $\sim 10^{-5}$ to $\sim 10^{-3}$ seconds on an AMD Athlon 1.1 GHz CPU and an UltraSparc II 400 MHz CPU to calculate the deflection and magnification with fractional calculational errors of 10^{-5} or smaller for any ellipticity $\epsilon \lesssim 0.7$.

Based on the present results, it is only around the break radius (specifically for $0.7r_b/\sqrt{1+e} \lesssim r \lesssim 1.3r_b/\sqrt{1-e}$) that the calculation is limited, mainly due to difficulty in efficiently evaluating an $I^{(1)}$-function given by equation (30a). There are two limitations: First, for ellipticities larger than $\epsilon \sim 0.7$, smallest possible fractional calculational errors are $\sim 10^{-5}$. Second, the calculation slows down significantly compared with calculations outside the above range of radius; on the same machines and with the same limits on the fractional calculational error and ellipticity given above execution time is typically $\sim 0.5 \times 10^{-3}$ to $\sim 10^{-2}$ seconds. These limitations, although not very significant, can be reduced if a new $I^{(1)}$-function simpler than equation (30a) is obtained in the future.

To summarize, the present Fourier series solutions to the deflection and magnification

---

[6]The calculational speed in the Fourier series approach (for any model) depends on the mass ellipticity for any value of r/r_b. Execution time increases approximately linearly with ellipticity for a fixed series truncation criterion.

by the 'cusp' model improve the calculational speed by factors of tens for a large portion of the parameter space of the model, compared to numerical integrations. For radii close to the break radius, the Fourier method still appears to be a few times faster than the numerical integration method for relatively small ellipticities (e.g. $\epsilon \lesssim 0.5$). The present Fourier series solutions for the 'cusp' model can also be used to calculate very accurate the deflection and magnification with fractional calculational errors of 10^{-6} or smaller, for the entire parameter space but the small portion of the parameter space specified above. Therefore, for practical applications of the 'cusp' model in which one is mainly interested in calculating the deflection and magnification at radii not too close to the break radius, the present Fourier series solutions greatly reduce his/her numerical efforts.

Numerical implementation of the I-functions of the 'cusp' model given in this paper is relatively straightforward except for two of them which are discussed in Appendix C. However, for the convenience of the lensing community well-tested codes both for the 'cusp' [eqs. (2a,b)] and 'sple' [eq. (4)] models are made publicly available on the following web page **http://www.jb.man.ac.uk/~chae**. The codes for the 'cusp' and 'sple' models are given the names of 'cuspFS' and 'spleFS' respectively, where 'FS' stands for Fourier series.

The remarkable calculational speed for a chosen small calculational error and the easy control over the calculational error (or, speed), which the Fourier series solutions both to the 'sple' and 'cusp' models demonstrate, suggest that the Fourier series method is the most efficient method for calculating the deflection and magnification for a general class of mass distributions. In particular, for any surface mass density which is a function of ζ [eq. (3)], one only needs to calculate the I-functions of the model [eqs. (A1) through (A5)] to calculate both the deflection and magnification. As shown in CKT and in this paper, the I-functions both of the 'sple' and 'cusp' models are well-defined, relatively simple converging series. It would be expected that for other classes of mass distributions which have not yet been explored, similar relatively simple expressions for the I-functions exist. For example, a generalized softened power-law ellipsoid with an exponentially declining envelope given by

$$\rho(R) = \frac{\rho_0}{(1 + R^2)^{\nu/2}} \exp(-R^2/R_t^2), \tag{37}$$

where $R^2 = (X/a)^2 + (Y/b)^2 + (Z/c)^2$, has the projected surface mass density given by the following equation:

$$\kappa(\zeta) = \kappa_0 (1 + \zeta^2)^{(1-\nu)/2} \exp(-\zeta^2/R_t^2) \Psi\left(\frac{1}{2}, \frac{3}{2} - \frac{\nu}{2}; \frac{1 + \zeta^2}{R_t^2}\right), \tag{38}$$

where $\Psi(a, b; z)$ is a confluent hypergeometric function and ζ is given by equation (3). Equation (38) can be expanded in terms of 'sple' models, and thus the I-functions of the model can be calculated in a straightforward manner.

Before we conclude this paper, below we discuss astrophysical motivations and aspects of using the 'cusp' double power-law model family as lens models, which have not been emphasized in previous work on gravitational lensing. For the 'cusp' model given by equation (1), the break radius and the outer power-law index have been used as defining outer envelopes of systems of stellar objects (e.g. Jaffe 1983, Hernquist 1990, Dehnen 1993, Tremaine et al. 1994, Zhao 1996, Muñoz et al. 2001), $\nu_o = 4$ being a preferred choice for the outer power-law index. Thus, the break radius would be a scale size of the system and the inner power-law index would represent an overall radial profile within the scale size. However, we also would like to use the break radius as a means to allow the radial profile of the lens to vary from a shallower inner profile, which is on average shallower than isothermal, to a steeper outer profile which is on average similar to isothermal and not steeper than $\nu_o = 3$ in general. This would be supported both by observational evidence and theoretical arguments. First, rotation curves of spiral galaxies, dwarf galaxies, and low surface brightness galaxies generally rise in the inner regions of the galaxies and are nearly flat beyond a turnover radius (e.g. Kravtsov et al. 1998; Rubin, Waterman, & Kenney 1999; Sofue et al. 1999; van den Bosch et al. 2000; Swaters, Madore, & Trewhella 2000; de Blok et al. 2001). Rotation curves of small samples of ellipticals/S0 galaxies and blue compact galaxies show similar radial behaviors (Bertola & Capaccioli 1975, 1977, 1978, Rubin, Peterson, & Ford 1980; Östlin et al. 1999). These rotation curves imply shallower-than-isothermal inner profiles and steeper outer profiles which are similar to isothermal but generally not steeper than $\nu_o = 3$. Second, *Hubble Space Telescope* luminosity profiles of the centers of early-type galaxies show shallower-than-isothermal profiles (e.g. Rest et al. 2001; Faber et al. 1997; Gebhardt et al. 1996) virtually for all the galaxies imaged. These observations imply that, at least at the very centers of the galaxies where dark matter contribution to the total mass density is minimal, mass profiles are shallower than isothermal. Since we do not expect such shallow profiles to extend to large radii, varying radial profiles with relatively small break radii appear to be inevitable. Third, numerical simulations of halo formation in cold dark matter models predict mass distributions with varying radial profiles (e.g. Navarro, Frenk, White 1997; Moore et al. 1998; Jing & Suto 2000; Ghigna et al. 2000) with shallower-than-isothermal inner profiles and an outer profile of $\nu_o = 3$. Moreover, mass distributions in the inner regions of the galaxies are expected to be modified due to the effects of the baryons (e.g. Blumenthal et al. 1986; Mo, Mao, & White 1998; Cole et al. 2000; Gonzalez et al. 2000; Kochanek & White 2001). It is thus of considerable interest to use gravitational lenses to determine the inner profiles, break radii, and outer profiles (and eventually truncation radii, i.e., halo sizes) for lensing (mostly early-type) galaxies. For this use of the outer power-law radial index, equation (1) will, in general, have a diverging total mass without a well-defined extent of the galaxy, although it would not be an issue in applications of the model to gravitational strong lensing, in which the extent of the dark matter halo is assumed to be much larger than the critical radius of

the lens. In order for the 'cusp' model [eq. (1)] with $\nu_o \leq 3$ to have a well-defined extent and converging total mass, we could perhaps introduce an envelope declining steeper than $\nu_o = 3$. In practice, it is likely that such a model is difficult to deal with mathematically. However, we could mimic an envelope using a subtraction term; in other words, we could use the following modified version of equation (1):

$$\rho(R) = \rho_0 R^{-\nu_i}[(1 + R^2)^{-(\nu_o - \nu_i)/2} - (R_t^2 + R^2)^{-(\nu_o - \nu_i)/2}]. \tag{39}$$

For large radius, the above model [eq. (39)] can be approximated by $\rho(R) \propto R^{-(\nu_o+2)}$, which permits a finite total mass for any $\nu_o > 1$.

Finally, it is worth noting that in this paper we limited ourselves to the families of two power-law mass distributions given by equations (1) and (39) mainly because of straightforward mathematical tractabilities of the models. The following model [see Hernquist (1990) and Zhao (1996)], which is the generalized version of equation (39), would eventually be of use in future gravitational lens studies:

$$\rho(R) = \rho_0 R^{-\nu_i}[(1 + R^\gamma)^{-(\nu_o - \nu_i)/\gamma} - (R_t^\gamma + R^\gamma)^{-(\nu_o - \nu_i)/\gamma}], \tag{40}$$

where parameter γ represents the "sharpness" of the break. However, until practical solutions to lensing by the model given by equation (40) are obtained and observational data sensitive enough to constrain parameter γ necessitate use of the model, equation (39) corresponding to $\gamma = 2$ in equation (40) will be useful to study galactic radial profiles by applying the model to gravitational lenses with the help of the calculational method presented in this paper.

A. Calculation of *I*-functions of 'Nuker' Model

The *I*-functions of any surface mass density which is a function of ζ [eq. (3)] are defined as follows:

$$I^{(0)}(r) = \frac{1}{r\pi} \int_0^{2\pi} d\phi' \int_0^r dr' r' \kappa(\zeta), \tag{A1}$$

$$I^{(0')}(r) = \frac{r}{\pi} \int_0^{2\pi} d\phi' \kappa(\zeta), \tag{A2}$$

$$I_{2m}^{(1)}(r) = \frac{r^{-1-2m}}{\pi \cos 2m\phi_0} B_{2m}(r) = \frac{r^{-1-2m}}{\pi \cos 2m\phi_0} \int_0^{2\pi} d\phi' \cos 2m\phi' \int_0^r dr' r'^{2m+1} \kappa(\zeta), \tag{A3}$$

$$I_{2m}^{(2)}(r) = \frac{r^{-1+2m}}{\pi \cos 2m\phi_0} D_{2m}(r) = \frac{r^{-1+2m}}{\pi \cos 2m\phi_0} \int_0^{2\pi} d\phi' \cos 2m\phi' \int_r^\infty dr' r'^{-2m+1} \kappa(\zeta), \tag{A4}$$

$$I_{2m}^{(3)}(r) = \frac{r}{2\pi \cos 2m\phi_0} \int_0^{2\pi} d\phi' \cos 2m\phi' \kappa(\zeta), \tag{A5}$$

where $\zeta = \zeta(r', \phi') = (r'/r_b)[1 + e \cos 2(\phi' - \phi_0)]^{1/2}$ and $m = 1, 2, 3, \ldots$[7]

Now for the 'Nuker' model [eq. (5)] for non-integer values of λ we have

$$
\begin{aligned}
B_{2m}(r) &= \int_0^{2\pi} d\phi' \cos 2m\phi' \int_0^r dr' r'^{2m+1} \kappa^{\text{Nuker}}(r', \phi') \\
&= \int_0^{2\pi} d\phi' \cos 2m\phi' \int_0^r dr' r'^{2m+1} \frac{[r'^2 g(\phi')]^\lambda}{[1 + r'^2 g(\phi')]^{\mu+1}} \\
&= \frac{r^{2(m+1)}}{2(\lambda + m + 1)} \int_0^{2\pi} d\phi' \cos 2m\phi' [g(\phi')r^2]^\lambda \\
&\quad \times F[\mu + 1, \lambda + m + 1; \lambda + m + 2; -g(\phi')r^2] \\
&= \frac{r^{2(\lambda+m+1)}}{2(\lambda + m + 1)} \sum_{n=0}^\infty \frac{(-1)^n (\mu + 1)_n (\lambda + m + 1)_n}{(\lambda + m + 2)_n n!} \\
&\quad \times r^{2n} \int_0^{2\pi} d\phi' \cos 2m\phi' [g(\phi')]^{\lambda+n}
\end{aligned}
\tag{A6}
$$

where $g(\phi') \equiv P + Q \cos 2(\phi' - \phi_0) \equiv r_b^{-2}[1 + e \cos 2(\phi' - \phi_0)]$ $(e, Q \geq 0)$ as defined in CKT. The integral in the last line of equation (A6) can be evaluated, using equation (B3) of CKT, as follows

$$\int_0^{2\pi} d\phi' \cos 2m\phi' [g(\phi')]^{\lambda+n} = 2\pi(-1)^m \cos(2m\phi_0) \frac{(P^2 - Q^2)^{(\lambda+n)/2}}{(1 + \lambda + n)_m} P_{\lambda+n}^m \left(\frac{P}{\sqrt{P^2 - Q^2}} \right). \tag{A7}$$

Now using equation (A4) of CKT and noting the definition of $I_{2m}^{(1)}(r)$ above [eq. (A3)], equation (17) for $i = 1$ is obtained.

Similarly we have

$$D_{2m}(r) = \int_0^{2\pi} d\phi' \cos 2m\phi' \int_r^\infty dr' r'^{-2m+1} \frac{[r'^2 g(\phi')]^\lambda}{[1 + r'^2 g(\phi')]^{\mu+1}}$$

---

[7]Notice that $d[I^{(0)}(r)]/dr = [-I^{(0)}(r) + I^{(0')}(r)]/r$, and similarly for $I_{2m}^{(1)}(r)$ and $I_{2m}^{(2)}(r)$.

$$= \frac{r^{-2(m-\lambda+\mu)}}{2(m-\lambda+\mu)} \int_0^{2\pi} d\phi' \cos 2m\phi' [g(\phi')]^{\lambda-\mu-1}$$

$$\times F\left[\mu+1, m-\lambda+\mu; m-\lambda+\mu+1; -\frac{1}{g(\phi')r^2}\right]. \tag{A8}$$

The hypergeometic function in equation (A8) can be re-written, via a linear transformation of hypergeometric functions (e.g. Gradshteyn & Ryzhik 1994), as follows

$$F\left[\mu+1, m-\lambda+\mu; m-\lambda+\mu+1; -\frac{1}{g(\phi')r^2}\right] =$$

$$\frac{\Gamma(m-\lambda+\mu+1)\Gamma(m-\lambda-1)}{\Gamma(m-\lambda+\mu)\Gamma(m-\lambda)}[g(\phi')r^2]^{\mu+1}$$

$$\times F[\mu+1, -m+\lambda+1; -m+\lambda+2; -g(\phi')r^2]$$

$$+\frac{\Gamma(m-\lambda+\mu+1)\Gamma(-m+\lambda+1)}{\Gamma(\mu+1)}[g(\phi')r^2]^{m-\lambda+\mu} \tag{A9}$$

Substituting equation (A9) into equation (A8) and integrating over ϕ', we find that the second term in equation (A9) vanishes. The rest of the calculation is very similar to that for $B_{2m}(r)$ and we obtain equation (17) for $i = 2$. The function $I_{2m}^{\text{Nuker(3)}}(r; \lambda, \mu)$ [eq. (17) for $i = 3$] can also be obtained in a very similar way from the definition [eq. (A5)].

B.   Revisiting 'SPLE' Model: Some New Expressions for I-Functions

As described in §3, I-functions of the 'cusp' model [eqs. (2a,b)] are calculated using those of the 'sple' model [eq. (4)] as well as those of the 'Nuker' model [eq. (5)] since the 'cusp' model can be expanded in terms of the 'sple' and 'Nuker' models. Here we consider some new expressions for the I-functions of the 'sple' model, which are useful for calculating I-functions of the 'cusp' model.

The I-functions of the 'sple' model obtained in CKT converge very quickly either for large or small radii compared with the core radius, which is denoted by r_b in equation (4). However, neither expression for the $I^{(1)}$-function given in CKT converges sufficiently fast to be useful for calculating the $I^{(1)}$-function of the 'cusp' model for $r \sim r_b$, although the $I^{(2)}$-function given in CKT is still useful.

We now obtain new expressions for the $I^{(1)}$- and $I^{(2)}$-functions of the 'sple' model which converge very quickly for $r \sim r_b$. This is done via the following expansion of the 'sple' model,

$$\kappa^{\text{sple}}(r, \phi) = \left\{1 + \left(\frac{r}{r_b}\right)^2 [1 + e\cos 2(\phi - \phi_0)]\right\}^{-(\mu+1)}$$

$$= \left[1 + \left(\frac{r}{r_b}\right)^2\right]^{-(\mu+1)} \left\{1 + \frac{(\frac{r}{r_b})^2}{1 + (\frac{r}{r_b})^2} e \cos 2(\phi - \phi_0)\right\}^{-(\mu+1)}$$

$$= \sum_{k=0}^{\infty} \frac{(-1)^k (\mu+1)_k}{k!} \frac{(\frac{r}{r_b})^{2k}}{[1 + (\frac{r}{r_b})^2]^{k+\mu+1}} e^k [\cos 2(\phi - \phi_0)]^k. \tag{B1}$$

After straightforward algebra using equation (B1), we obtain

$$I_{2m}^{\text{sple}(1)}(r; \mu) = \frac{r}{\pi}(-te)^m (1-t)^{\mu+1} \sum_{k=0}^{\infty} \frac{\Gamma(k+m+\mu+1)}{\Gamma(\mu+1)\Gamma(k+m+1)} \frac{1+(-1)^k}{k+2m+1}$$

$$\times (te)^k \text{Icos}(k+m, m) F(k+m+\mu+1, 1; k+2m+2; t), \tag{B2}$$

$$I_{2m}^{\text{sple}(2)}(r; \mu) = \frac{r}{\pi}(-te)^m (1-t)^{\mu+1} \frac{1}{(\mu+m)t} \sum_{k=0}^{\infty} \frac{\Gamma(k+m+\mu+1)}{\Gamma(\mu+1)\Gamma(k+m+1)}$$

$$\times [1+(-1)^k] e^k \text{Icos}(k+m, m) F(-k, m+\mu; m+\mu+1; 1-t), \tag{B3}$$

where t and $\text{Icos}(k, m)$ are given in equation (30a).

C. Prescriptions for Evaluating Non-trivial I-Functions

All the I-functions obtained in CKT and in this paper are well-defined converging series. Numerical implementation of them is relatively straightforward. However, a few of them need to be dealt with carefully due to their non-trivially converging properties for part of the parameter space. Below we discuss these non-trivial I-functions and give some mathematical prescriptions which are useful for numerically evaluating them.

Let us first consider the $I^{(2)}$-function of the 'sple' model obtained in CKT, which is given by

$$I_{2m}^{\text{sple}(2)}(r; \mu) = h(r)[-\sqrt{\varepsilon_1(r)}]^m [\varepsilon_2(r)]^\mu \frac{\Gamma(m+\mu)}{\Gamma(m+1)\Gamma(\mu+1)}$$

$$\times \sum_{k=0}^{\infty} [\varepsilon_2(r)]^k F[m-k-\mu, -k-\mu; m+1; \varepsilon_1(r)]. \tag{C1}$$

While this function converges quickly for $r \gtrsim r_b$ since $\varepsilon_2(r) \to 0$ for $r \gtrsim r_b$, it converges slowly for $r < r_b$ especially as $r/r_b \to 0$. The latter case (i.e. $r < r_b$) is irrelevant in most applications of the 'sple' model since observed images are normally found well outside any lens core radii. However, as can be seen in equations (14) and (23), evaluation of the $I^{(2)}$-function of the 'cusp' model for $r < r_b$ requires evaluation of the $I^{(2)}$-function of the 'sple' model [eq. (C1)] for $r < r_b$ multiple times. It is thus important to evaluate equation (C1)

fast and accurately for $r < r_b$. Fast and accurate evaluation of equation (C1) can be done by expanding the hypergeomtric function and noting that infinite sum for each term in the hypergeometric function can be broken down to terms including

$$S_n(x) = \sum_{k=1}^{\infty} k^n x^k \quad (n = 0, 1, 2, \ldots), \tag{C2}$$

which can be analytically evaluated using the following recursion relation

$$S_n(x) = \frac{-1}{1-x} \sum_{k=0}^{n-1} (-1)^{n-k} \binom{n}{k} S_k(x) \tag{C3}$$

and $S_0(x) = x/(1-x)$.

As noted in §3, the $I^{(1)}$-function given by equation (30a) of the 'cusp' model requires a special treatment. Equation (30a), which was obtained by substituting equation (B2) into equation (26), has a slow convergence over l although its convergence over k is fast. This slow convergence problem cannot be overcome using equation (27). In fact, the $I^{(1)}$-function given by equation (30a), which is needed for $r \sim r_b$, is the only true slowly converging series out of all the I-functions given in CKT and in this paper. Let the l-th term in the infinite sum over l in equation (30a) be denoted by u_l, i.e.,

$$u_l = D_l(\nu_i, \nu_o)(1-t)^l \frac{\Gamma(l+k+m+\mu+1)}{\Gamma(l+\mu+1)} F(l+k+m+\mu+1, 1; k+2m+2; t). \tag{C4}$$

As $l \to \infty$, we have $u_l/u_{l+1} \to 1$ and

$$l\left(\frac{u_l}{u_{l+1}} - 1\right) \to a > 1, \tag{C5}$$

which implies

$$u_l \to \frac{l-1}{a+l-1} u_{l-1}. \tag{C6}$$

Equation (C6) can then be used to approximate the summation after $l_0 \gg 1$ as follows,

$$\sum_{l=l_0}^{\infty} u_l \to u_1 \frac{a}{a-1} - \sum_{l=1}^{l_0-1} u_l. \tag{C7}$$

REFERENCES

Abramowitz, M., & Stegun, I. 1964, Handbook of Mathematical Functions with Formulas, Graphs, and Mathematical Tables (New York: Dover Publications)

Barkana, R. 1998, ApJ, 502, 531

Bertola, F., Capaccioli, M. 1975, ApJ, 200, 439

Bertola, F., Capaccioli, M. 1977, ApJ, 211, 697

Bertola, F., Capaccioli, M. 1978, ApJ, 219, 404

Blumenthal, G.R., Faber, S.M., Flores, R., Primack, J.R. 1986, ApJ, 301, 27

Chae, K.-H., Khersonsky, V.K., & Turnshek, D.A. 1998, ApJ, 506, 80 (CKT)

Cole, S., Lacey, C.G., Baugh, C.M., & Frenk, C.S. 2000, MNRAS, 319, 168

de Blok, W.J.G., McGaugh, S.S., Bosma, A., Rubin, V.C. 2001, ApJ, 552, L23

Dehnen, W., 1993, MNRAS, 265, 250

Evans, N.W., & Wilkinson, M.I. 1998, MNRAS, 296, 800

Faber, S.M., Tremaine, S., Ajhar, E.A., Byun, Y.-I., Dressler, A., Gebhardt, K., Grillmair, C., Kormendy, J., Lauer, T.R., & Richstone, D. 1997, AJ, 114, 1771

Gebhardt, K., Richstone, D., Ajhar, E.A., Lauer, T.R., Byun, Y.-I., Kormendy, J., Dressler, A., Faber, S.M., Grillmair, C., & Tremaine, S. 1996, AJ, 112, 105

Ghigna, S., Moore, B., Governato, F., Lake, G., Quinn, T., Stadel, J. 2000, ApJ, 544, 616

Gonzalez, A.H., et al. 2000, ApJ, 528, 145

Gradshteyn, I.S., & Ryzhik, I.M. 1994, Table of Integrals, Series, and Products (5th ed.; Academic Press)

Hernquist, L. 1990, ApJ, 356, 359

Jaffe, W. 1983, MNRAS, 202, 995

Jing, Y.P., Suto, Y. 2000, ApJ, 529, L69

Keeton, C.R. 2001, ApJ, in press (astro-ph/0102340)

Kochanek, C.S., & White, M. 2001, ApJ, 559, 531

Kravtsov, A.V., Klypin, A.A., Bullock, J.S., & Primack, J.R. 1998, ApJ, 502, 48

Lauer, T.R., Ajhar, E.A., Byun, Y.-I., Dressler, A., Faber, S.M., Grillmair, C., Kormendy, J., Richstone, D., & Tremaine, S. 1995, AJ, 110, 2622

Mo, H.J., Mao, S., White, S.D.M. 1998, MNRAS, 295, 319

Moore, B., Governato, F., Quinn, T., Stadel, J., & Lake G. 1998, ApJ, 499, L5

Muñoz, J.A., Kochanek, C.S., & Keeton, C.R. 2001, ApJ, 558, 657

Navarro, J.F., Frenk, C.S., & White, S.D.W. 1997, ApJ, 490, 493

Östlin, G., Amram, P., Masegosa, J., Bergvall, N., Boulesteix, J. 1999, A&AS, 137, 419

Refsdal, S. 1964, MNRAS, 128, 307

Rest, A., van den Bosch, F.C., Jaffe, W., Tran, H., Tsvetanov, Z., Ford, H.C., Davies, J., & Schafer, J. 2001, AJ, 121, 2431

Rubin, V.C., Peterson, C.J., Ford, W.K., Jr. 1980, ApJ, 239, 50

Rubin, V.C., Waterman, A.H., & Kenney, J.D.P. 1999, AJ, 118, 236

Schneider, P., Ehlers, J., & Falco, E.E. 1992, Gravitational Lenses (New York: Springer-Verlag)

Sofue, Y., Tutui, Y., Honma, M., Tomita, A., Takamiya, T., Koda, J., & Takeda, Y. 1999, ApJ, 523, 136

Swaters, R.A., Madore, B.F., Trewhella, M. 2000, ApJ, 531, L107

Tremaine, S., et al. 1994, AJ, 107, 634

van den Bosch, F.C., Robertson, B.E., Dalcanton, J.J., & de Blok, W.J.G. 2000, AJ, 119, 1579

Zhao, H.S. 1996, MNRAS, 278, 488

Table 1
Summary of Mass Models

name	meaning	projected density	3-dimensional density
SPLE	softened power-law ellipsoid	Equation (4)	$\rho(R) = \rho_0(1 + R^2)^{-\nu/2}$ [1]
Nuker	"Nuker-law" model	Equation (5)	Not Applicable
Cusp	cusped two power-law model	Equations (2a,b)	Equation (1)

[1] See equation (1) for the definition of R.